

AD-A138 220

GENERALIZED CONVEXITY AND CONCAVITY PROPERTIES OF THE  
OPTIMAL VALUE FUNCT..(U) GEORGE WASHINGTON UNIV  
WASHINGTON DC INST FOR MANAGEMENT SCIE..

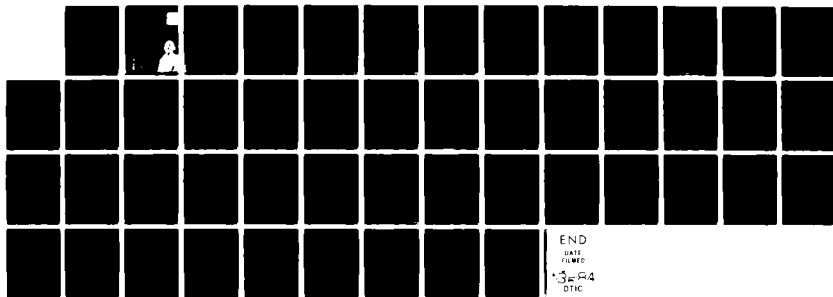
1/1

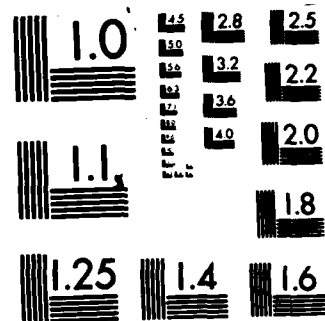
UNCLASSIFIED

A V FIACCO ET AL. 11 APR 83

F/G 12/1

NL





MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

12

REALIZED CONVEXITY AND CONCAVITY PROPERTIES  
THE OPTIMAL VALUE FUNCTION IN PARAMETRIC  
NONLINEAR PROGRAMMING

by

Anthony V. Fiacco  
Jerzy Kyparisis

THE  
GEORGE  
WASHINGTON  
UNIVERSITY

ADA138220

STUDENTS FACULTY STUDY R  
ESEARCH DEVELOPMENT FUT  
URE CAREER CREATIVITY CO  
MMUNITY LEADERSHIP TECH  
NOLOGY FRONTIER DESIGN  
ENGINEERING APP ENO  
GEORGE WASHINGTON UNIV

S FEB 23 1984  
A

02 21 071

INSTITUTE FOR MANAGEMENT  
SCIENCE AND ENGINEERING  
SCHOOL OF ENGINEERING  
AND APPLIED SCIENCE



GENERALIZED CONVEXITY AND CONCAVITY PROPERTIES  
OF THE OPTIMAL VALUE FUNCTION IN PARAMETRIC  
NONLINEAR PROGRAMMING

by

Anthony V. Fiacco  
Jerzy Kyparisis

RESOURCE DYNAMICS  
and  
NATIONAL SCIENCE FOUNDATION  
GWU/IMSE/Serial T-472/83  
11 April 1983

THE GEORGE WASHINGTON UNIVERSITY  
School of Engineering and Applied Science  
Washington, DC 20052  
Institute for Management Science and Engineering

Research Supported  
by  
Contract N00014-83-K-0215  
Project NR 047 624  
Office of Naval Research  
and  
Grant ECS-8201370  
National Science Foundation

This document has been approved for public sale  
and release; its distribution is unlimited.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER  GWU/IMSE/T-472/83	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle)  GENERALIZED CONVEXITY AND CONCAVITY PROPERTIES OF THE OPTIMAL VALUE FUNCTION IN PARAMETRIC NONLINEAR PROGRAMMING	5. TYPE OF REPORT & PERIOD COVERED  SCIENTIFIC	
7. AUTHOR(s)  ANTHONY V. FIACCO JERZY KYPARISIS	6. PERFORMING ORG. REPORT NUMBER  T-472	
9. PERFORMING ORGANIZATION NAME AND ADDRESS GEORGE WASHINGTON UNIVERSITY INSTITUTE FOR MANAGEMENT SCIENCE AND ENGINEERING WASHINGTON, DC 20052	8. CONTRACT OR GRANT NUMBER(s) N00014-83-K-0215 ECS-8201370	
11. CONTROLLING OFFICE NAME AND ADDRESS OFFICE OF NAVAL RESEARCH CODE 411 S&P ARLINGTON, VA 22217	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	12. REPORT DATE 11 April 1983	
	13. NUMBER OF PAGES 46	
	15. SECURITY CLASS. (of this report)  UNCLASSIFIED	
16. DISTRIBUTION STATEMENT (of this Report)  APPROVED FOR PUBLIC SALE AND RELEASE; DISTRIBUTION UNLIMITED.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
PARAMETRIC NLP OPTIMAL VALUE FUNCTION CONVEXITY CONCAVITY	GENERALIZED CONVEXITY GENERALIZED CONCAVITY SOLUTION POINT-TO-SET MAP QUASICONVEXITY	
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)		
<p>In this paper we consider generalized convexity and concavity properties of the optimal value function <math>f^*</math> for the general parametric optimization problem <math>P(e)</math> of the form <math>\min_x f(x,e)</math> s.t. <math>x \in R(e)</math>. Many results on convexity and concavity characterizations of <math>f^*</math> were presented by the authors in a previous paper. Such properties of <math>f^*</math> and the solution set map <math>S^*</math> form</p> <p style="text-align: right;">(continued)</p>		

DD FORM 1473

1 JAN 73

EDITION OF 1 NOV 68 IS OBSOLETE  
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

20. Abstract (cont'd)

an important part of the theoretical basis for sensitivity, stability and parametric analysis in mathematical optimization. We give sufficient conditions for several types of generalized convexity and concavity of  $f^*$ , in terms of respective generalized convexity and concavity assumptions on  $f$  and convexity and concavity assumptions on the feasible region point-to-set map  $R$ . Specializations of these results to the general parametric inequality-equality constrained nonlinear programming problem and its right-hand-side version are provided.

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

- a -

THE GEORGE WASHINGTON UNIVERSITY  
School of Engineering and Applied Science  
Washington, DC 20052  
Institute for Management Science and Engineering

GENERALIZED CONVEXITY AND CONCAVITY PROPERTIES  
OF THE OPTIMAL VALUE FUNCTION IN PARAMETRIC  
NONLINEAR PROGRAMMING

by

Anthony V. Fiacco  
Jerzy Kyparisis

Abstract  
of  
RESOURCE DYNAMICS  
and  
NATIONAL SCIENCE FOUNDATION  
GWU/IMSE/Serial T-472/83  
11 April 1983

*epsilon*

*x sub*

In this paper we consider generalized convexity and concavity properties of the optimal value function  $f^*$  for the general parametric optimization problem  $P(e)$  of the form  $\min_x f(x,e)$  s.t.  $x \in R(e)$ . Many results on convexity and concavity characterizations of  $f^*$  were presented by the authors in a previous paper. Such properties of  $f^*$  and the solution set map  $S^*$  form an important part of the theoretical basis for sensitivity, stability and parametric analysis in mathematical optimization. We give sufficient conditions for several types of generalized convexity and concavity of  $f^*$ , in terms of respective generalized convexity and concavity assumptions on  $f$  and convexity and concavity assumptions on the feasible region point-to-set map  $R$ . Specializations of these results to the general parametric inequality-equality constrained nonlinear programming problem and its right-hand-side version are provided.

*The authors*

Research Supported  
by  
Contract N00014-83-K-0215  
Project NR 047 624  
Office of Naval Research  
and  
Grant ECS-8201370  
National Science Foundation

*11*

Div. A-1

## Table of Contents

	Page
1. Introduction . . . . .	1
2. Basic (generalized) convexity and concavity notions . . . . .	3
3. Quasiconvexity of the optimal value function . . . . .	6
4. Quasiconcavity and quasimonotonicity of the optimal value function . . . . .	17
5. Pseudoconvexity and pseudoconcavity of the optimal value function . . . . .	31
6. Concluding remarks . . . . .	36
References . . . . .	38



THE GEORGE WASHINGTON UNIVERSITY  
 School of Engineering and Applied Science  
 Washington, DC 20052  
 Institute for Management Science and Engineering

GENERALIZED CONVEXITY AND CONCAVITY PROPERTIES  
 OF THE OPTIMAL VALUE FUNCTION IN PARAMETRIC  
 NONLINEAR PROGRAMMING

by

Anthony V. Fiacco  
 Jerzy Kyparisis

RESOURCE DYNAMICS  
 and  
 NATIONAL SCIENCE FOUNDATION  
 GWU/IMSE/Serial T-472/83  
 11 April 1983

1. Introduction

We study a general parametric optimization problem of the form

$$\min f(x,e) \quad \text{s.t.} \quad x \in R(e) \quad P(e)$$

where  $f: E^n \times E^k \rightarrow E^1$  and  $R$  is a point-to-set map assigning to points in  $E^k$  subsets (possibly empty) of  $E^n$ . Several specializations of this problem are also considered. We shall be primarily interested in the characterization of generalized convexity and concavity properties of the optimal value function  $f^*$  of the problem  $P(e)$ , defined as

$$f^*(e) = \begin{cases} \inf_x \{f(x,e) | x \in R(e)\} & , \text{ if } R(e) \neq \emptyset \\ +\infty & , \text{ if } R(e) = \emptyset . \end{cases}$$

Many convexity and concavity properties of  $f^*$  were presented by the authors in a previous paper (Fiacco and Kyparisis (1982)). Here we continue this line of investigation and give sufficient conditions for additional generalized convexity and concavity properties of  $f^*$ , e.g., quasiconvexity, quasiconcavity, quasimonotonicity, pseudoconvexity and

pseudoconcavity, given various assumptions on  $f$  and the feasible set map  $R$ .

Convexity and concavity properties as well as other principal properties of the optimal value function  $f^*$  such as continuity, differentiability, etc. have attracted much interest since they form a theoretical basis for sensitivity, stability and parametric analysis in nonlinear optimization. We refer the interested reader to the recent monographs by Rockafellar (1974), Brosowski (1982), Bank et.al. (1982) and Fiacco (1983). This paper presents many results, almost all of which are new, in a unified manner, utilizing a number of definitions introduced here as well as in the previous paper by the authors. Readers less familiar with the various concepts of (generalized) convexity and concavity may consult the books by Mangasarian (1969) and Ortega and Rheinboldt (1970), or a recent survey by Avriel et. al. (1981). For the notions of convex analysis the reader is referred to Rockafellar (1970), and for those of topology and point-to-set map theory to Berge (1963).

We briefly specify the several programs studied in this paper. A special case of  $P(e)$  is the parametric optimization problem

$$\min f(x) \quad \text{s.t.} \quad x \in R(e) \quad P'(e)$$

The parametric nonlinear programming (NLP) problem  $P_3(e)$  is of the form  $P(e)$  with the map  $R$  defined by

$$R(e) = \{x \in M \mid g_i(x,e) \geq 0, i=1,\dots,m, h_j(x,e) = 0, j=1,\dots,p\}$$

where  $M \subset E^n$ ,  $g_i = E^n \times E^k \rightarrow E^1, i = 1,\dots,m$ ,  $h_j = E^n \times E^k \rightarrow E^1, j=1,\dots,p$ .

The general right-hand-side (grhs) NLP problem  $P_2(e)$  is also of the form  $P(e)$  with  $R$  given by

$$R(e) = \{x \in M \mid g_i(x) \geq e_i, i=1,\dots,m, h_j(x) = e_{m+j}, j=1,\dots,p\}.$$

Note that  $P_2(e)$  differs from the standard rhs NLP problem for which

$f(x,e) = f(x)$ . Furthermore, a special problem  $P_1(e)$  is of the form  $P(e)$  with  $R$  given by  $R(e) = M$ .

Finally, we make a few comments on possible extensions and applications of the results presented here. Most of the results can be easily extended to more general spaces, e.g., real vector or Banach spaces, without any change in the proofs, and applied to abstract problems of optimal control, mathematical economics, etc. These results can also be applied and extended for various specific classes of problems of nonlinear programming, e.g., geometric programs, separable programs, etc. by taking advantage of the special structure of these problems. Computation of simple upper and lower parametric bounds on  $f^*$  is another possible application which should be exploited.

## 2. Basic (generalized) convexity and concavity notions

The material of this section is adapted from Fiacco and Kyparisis (1982). We state the definitions and several basic results for completeness, since we shall use them extensively in the subsequent sections. An (extended) function  $\phi: E^n \rightarrow E^1 \cup \{-\infty, \infty\}$  is called "quasiconvex" on a convex set  $M \subset E^n$  if for all  $x_1, x_2 \in M$  and  $\lambda \in (0,1)$ ,

$$\phi(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\phi(x_1), \phi(x_2)\},$$

and is called "quasiconcave" on  $M$  (see Fenchel (1953)) if  $-\phi$  is quasiconvex on  $M$ . Also,  $\phi$  is called "quasimonotonic" on  $M$  (Martos (1967)) if it is both quasiconvex and quasiconcave on  $M$ .

The following definition is well known. The point-to-set map  $R: E^k \rightarrow E^n$  is called "convex" on a convex set  $S \subset E^k$  if, for all  $e_1, e_2 \in S$  and  $\lambda \in (0,1)$ ,

$$\lambda R(e_1) + (1-\lambda) R(e_2) \subset R(\lambda e_1 + (1-\lambda)e_2).$$

The next two results state sufficient conditions for the convexity of the point-to-set map  $R$ .

Proposition 2.1. Consider the parametric NLP problem  $P_3(e)$ . If  $\{g_i\}$  are jointly quasiconcave on  $M \times S$ ,  $\{h_j\}$  are jointly quasimonotonic on  $M \times S$  and  $M$  and  $S$  are convex sets, then  $R$  given by  $R(e) = \{x \in M \mid g_i(x, e) \geq 0, i=1, \dots, m, h_j(x, e) = 0, j=1, \dots, p\}$  is convex on  $S$ .

Proposition 2.2. Consider the grhs parametric NLP problem  $P_2(e)$ . If  $\{g_i\}$  are concave on  $M$ ,  $\{h_j\}$  are affine on  $M$  and  $M$  is convex, then  $R$  given by  $R(e) = \{x \in M \mid g_i(x) \geq e_i, i=1, \dots, m, h_j(x) = e_m + e_j, j=1, \dots, p\}$  is convex.

The following extends the definition of convexity for point-to-set maps. The point-to-set map  $R: E^k \rightarrow E^n$  is called "essentially convex" on a convex set  $S \subset E^k$  if for all  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$  and  $\lambda \in (0, 1)$ ,

$$\lambda R(e_1) + (1-\lambda)R(e_2) \subset R(\lambda e_1 + (1-\lambda)e_2).$$

Define  $clA$  and  $conv A$  to be the closure and convex hull of the set  $A$ , respectively. The map  $R$  is called "closure convex" ("essentially closure convex") on  $S$  if the map  $clR$  given by  $clR(e) = cl(R(e))$  is convex (essentially convex) on  $S$ .  $R$  is also called "hull convex" ("essentially hull convex") on  $S$  if the map  $convR$  given by  $convR(e) = conv(R(e))$  is convex (essentially convex) on  $S$ .

We also consider concavity of the map  $R$ . The point-to-set map  $R: E^k \rightarrow E^n$  is called "concave" on a convex set  $S \subset E^k$  if, for all  $e_1, e_2 \in S$  and  $\lambda \in (0, 1)$ ,

$$R(\lambda e_1 + (1-\lambda)e_2) \subset \lambda R(e_1) + (1-\lambda) R(e_2) \text{ (see Tagawa (1978)).}$$

Similarly, as for convex maps we extend the last definition as follows. The map  $R$  is called "closure concave" on a convex set  $S$  if the map  $clR$  is concave on  $S$ . Also,  $R$  is called "hull concave" on  $S$  if the map  $convR$  is concave on  $S$ . We slightly strengthen the definition of concavity and call the map  $R$  "strictly concave" on  $S$  if for any  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$ ,  $\lambda \in (0, 1)$  and  $x \in R(\lambda e_1 + (1-\lambda)e_2)$  there exist

$x_1 \in R(e_1)$  and  $x_2 \in R(e_2)$  such that  $x_1 \neq x_2$  and  $x = \lambda x_1 + (1-\lambda) x_2$ . Also, we call  $R$  "strictly hull concave" on  $S$  if the map  $\text{conv}R$  is strictly concave on  $S$ .

Combining the previous definitions we call  $R$  "affine" ("essentially affine") (see Tagawa (1978) and Penot (1982)) on  $S$  if  $R$  is both convex (essentially convex) and concave on  $S$ . Further, we extend these notions and call  $R$  "hull affine" ("essentially hull affine") on  $S$  if the map  $\text{conv}R$  is affine (essentially affine) on  $S$ .

Finally, we consider the notion of homogeneity. An (extended) function  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  is called "positively homogeneous" on a cone  $K \subset E^r$  (see Rockafellar (1970)) if  $\phi(\lambda x) = \lambda \phi(x)$  for all  $x \in K$  and  $\lambda > 0$ . Note that if  $\phi$  is positively homogeneous on  $K$  and  $0 \in K$  then  $\phi(0) = 0$  if  $\phi(0)$  is finite. The point-to-set map  $R: E^k \rightarrow E^n$  is called "positively homogeneous" on a cone  $K \subset E^k$  (see Rockafellar (1967)) if  $R(\lambda e) = \lambda R(e)$  for all  $e \in K$  and  $\lambda > 0$ . The following results provide sufficient conditions for the positive homogeneity of the optimal value function  $f^*$ .

**Proposition 2.3.** Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly positively homogeneous on  $E^n \times K$ ,  $R$  is positively homogeneous on  $K$  and  $K$  is a cone, then  $f^*$  is positively homogeneous on  $K$ .

**Proposition 2.4.** Consider the parametric NLP problem  $P_3(e)$ . If  $f$ ,  $\{g_i\}$  and  $\{h_j\}$  are jointly positively homogeneous on  $M \times K$ , and  $M$  and  $K$  are cones, then  $f^*$  is positively homogeneous on  $K$ .

**Proposition 2.5.** Consider the special problem  $P_1(e)$ . If  $f$  is positively homogeneous in  $e$  on  $K$  for every  $x \in E^n$ ,  $R(e) = M$  for any  $e \in K$  and  $\lambda > 0$ ,  $M$  is arbitrary and  $K$  is a cone, then  $f^*$  is positively homogeneous on  $K$ .

The following definitions for the maps combine the definition of positive homogeneity with convexity and concavity as previously introduced in this section. A point-to-set map  $R: E^k \rightarrow E^n$  will be called homogeneous convex" ("homogeneous concave") on a convex cone  $K \subset E^k$  if  $R$  is both positively homogeneous and convex (concave) on  $K$ . Homogeneous convex maps were introduced and studied by Rockafellar (1967, 1970) who calls such maps "convex processes." Convex-valued homogeneous concave maps were introduced and studied by Ioffe (1979) under the name "fans."

### 3. Quasiconvexity of the optimal value function

Recall from Section 2 that  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  is called "quasiconvex" on a convex set  $M \subset E^r$  if for all  $x_1, x_2 \in M$  and  $\lambda \in (0,1)$

$$\phi(\lambda x_1 + (1-\lambda)x_2) \leq \max \{ \phi(x_1), \phi(x_2) \}.$$

Convexity and essential convexity of  $R$  was also defined in Section 2.

Proposition 3.1. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly quasiconvex on the set  $\{(x,e) | x \in R(e), e \in S\}$ ,  $R$  is essentially convex or convex on  $S$  and  $S$  is convex, then  $f^*$  is quasiconvex on  $S$ .

#### Proof

Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$  and  $\lambda \in (0,1)$ . By essential convexity of  $R$  and quasiconvexity of  $f$  we obtain

$$\begin{aligned} f^*(\lambda e_1 + (1-\lambda)e_2) &= \inf_{x \in R(\lambda e_1 + (1-\lambda)e_2)} f(x, \lambda e_1 + (1-\lambda)e_2) \\ &\leq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} f(\lambda x_1 + (1-\lambda)x_2, \lambda e_1 + (1-\lambda)e_2) \leq \\ &\leq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} \max \{ f(x_1, e_1), f(x_2, e_2) \} = \end{aligned}$$

$$= \max \left\{ \inf_{x_1 \in R(e_1)} f(x_1, e_1), \inf_{x_2 \in R(e_2)} f(x_2, e_2) \right\} =$$

$$= \max \{ f^*(e_1), f^*(e_2) \},$$

i.e.,  $f^*$  is quasiconvex on  $S$ .

Proposition 3.1 can be specialized to the NLP problems  $P_3(e)$  and  $P_2(e)$  using Propositions 2.1 and 2.2, respectively. We give these results without a proof. Recall from Section 2 that  $\phi$  is quasiconcave on  $M$  if  $-\phi$  is quasiconvex on  $M$  and  $\phi$  is quasimonotonic on  $M$  if it is both quasiconvex and quasiconcave on  $M$ .

Corollary 3.2. Consider the parametric NLP problem  $P_3(e)$ . If  $f$  is jointly quasiconvex on  $M \times S$ ,  $\{g_i\}$  are jointly quasiconcave (or concave) on  $M \times S$ ,  $\{h_j\}$  are jointly quasimonotonic (or affine) on  $M \times S$ , and  $M$  and  $S$  are convex sets, then  $f^*$  is quasiconvex on  $S$ .

Corollary 3.3. Consider the grhs parametric NLP problem  $P_2(e)$ . If  $f$  is jointly quasiconvex on  $M \times S$ ,  $\{g_i\}$  are concave on  $M$ ,  $\{h_j\}$  are affine on  $M$ , and  $M$  and  $S$  are convex, then  $f^*$  is quasiconvex on  $S$ .

Luenberger (1968) obtained the result of Corollary 3.3 in the special case when  $f(x, e) = f(x)$  and no equality constraints are present. The next two results extend Proposition 3.1.

Proposition 3.4. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly quasiconvex on the set  $\{(x, e) | x \in \text{cl}(R(e)), e \in S\}$  and upper semicontinuous in  $x$  on the set  $\text{cl}(R(S))$  for every  $e \in S$ ,  $R$  is essentially closure convex or closure convex on  $S$  and  $S$  is convex, then  $f^*$  is quasiconvex on  $S$ .

Proof

Define  $\bar{R}(e) = \text{cl}(R(e))$  for  $e \in S$  and denote  $\bar{f}^*(e) = \inf_x \{f(x, e) \mid x \in \bar{R}(e)\}$ . Since  $\bar{R}$  is essentially convex, by Proposition 3.1  $\bar{f}^*$  is quasiconvex on  $S$ . Thus, we need only show that  $\bar{f}^*(e) = f^*(e)$  for  $e \in S$ . Clearly  $f^*(e) \geq \bar{f}^*(e)$  for  $e \in S$ . Let  $\{x_n\} \subset \bar{R}(e)$ ,  $e \in S$  be such that  $\bar{f}^*(e) = \lim_{n \rightarrow \infty} f(x_n, e)$  (such a sequence exists by definition of infimum). Then for every  $n$  there is a sequence  $\{x_{nm}\} \subset R(e)$  such that  $x_n = \lim_{m \rightarrow \infty} x_{nm}$ . Hence, by upper semicontinuity of  $f$

$$f(x_n, e) = f(\lim_{m \rightarrow \infty} x_{nm}, e) \geq \limsup_{m \rightarrow \infty} f(x_{nm}, e) \geq f^*(e),$$

what implies that  $\bar{f}^*(e) \geq f^*(e)$ ,  $e \in S$ .

Remark 3.5. The above proof shows that if  $f$  is upper semicontinuous in  $x$  and  $R(e)$  is an arbitrary set, then

$$\bar{f}^*(e) = \inf_{x \in \text{cl}(R(e))} f(x, e) = \inf_{x \in R(e)} f(x, e) = f^*(e).$$

Proposition 3.6. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly quasiconvex on the set  $\{(x, e) \mid x \in \text{conv}(R(e)), e \in S\}$  and quasiconcave in  $x$  on the set  $\text{conv}(R(S))$  for every  $e \in S$ ,  $R$  is essentially hull convex or hull convex on  $S$  and  $S$  is convex, then  $f^*$  is quasiconvex on  $S$ .

Proof

Define  $\tilde{R}(e) = \text{conv}(R(e))$  for  $e \in S$ . Denote  $\tilde{f}^*(e) = \inf_x \{f(x, e) \mid x \in \tilde{R}(e)\}$ . Since  $\tilde{R}$  is essentially convex, by Proposition 3.1  $\tilde{f}^*$  is quasiconvex on  $S$ . We shall show that  $\tilde{f}^*(e) = f^*(e)$  for  $e \in S$ . Obviously,  $f^*(e) \geq \tilde{f}^*(e)$  for  $e \in S$ . Suppose now that  $e \in S$  and  $x \in \tilde{R}(e)$ . Then  $x = \lambda x_1 + (1-\lambda)x_2$  for some  $x_1, x_2 \in R(e)$  and  $\lambda \in (0, 1)$ . By quasiconcavity of  $f$



$$\begin{aligned}
 f(x, e) &= f(\lambda x_1 + (1-\lambda)x_2, e) \geq \min\{f(x_1, e), f(x_2, e)\} \\
 &\geq \inf_{x \in R(e)} f(x, e) = f^*(e) .
 \end{aligned}$$

This implies that  $\tilde{f}^*(e) = \inf_{x \in \tilde{R}(e)} f(x, e) \geq f^*(e)$  .

Remark 3.7. In the above proof we have basically shown that if  $f$  is quasiconcave in  $x$  and  $R(e)$  is arbitrary, then

$$\begin{aligned}
 \tilde{f}^*(e) &= \inf_{x \in \text{conv}(R(e))} f(x, e) &= \inf_{x \in R(e)} f(x, e) &= f^*(e) .
 \end{aligned}$$

We call  $R: E^k \rightarrow E^n$  a "quasiconvex" point-to-set map on a convex set  $S \subseteq E^k$  if  $R^{-1}(M) \cap S$  (where we define  $R^{-1}(M) = \{e \in E^k \mid R(e) \cap M \neq \emptyset\}$ , see Berge(1963)) is convex for any convex set  $M \subseteq E^n$ . It is easy to see that an (essentially) convex point-to-set map  $R$  on  $S$  is also quasiconvex on  $S$ , but not conversely. The next result strengthens Proposition 3.1 for the problem  $P'(e)$  .

Proposition 3.8. Consider the parametric optimization problem  $P'(e)$  . If  $f$  is quasiconvex on the set  $R(S)$ ,  $R$  is quasiconvex on  $S$  and  $S$  is convex, then  $f^*$  is quasiconvex on  $S$  .

#### Proof

It is enough to show that for any  $c \in E^1 \cup \{+\infty\}$ ,  $L_c^0 = \{e \in S \mid f^*(e) < c\}$  is convex. Let  $e_1, e_2 \in L_c^0$  and  $\lambda \in (0, 1)$  . Then  $f^*(e_1) < c$ ,  $f^*(e_2) < c$ , so for some  $x_1 \in R(e_1)$ ,  $x_2 \in R(e_2)$ ,  $f(x_1) < c$ ,  $f(x_2) < c$  . Thus,  $e_1, e_2 \in R^{-1}([x_1, x_2]) \cap S$  and by our assumption also  $\lambda e_1 + (1-\lambda)e_2 \in R^{-1}([x_1, x_2]) \cap S$  . This implies that for some  $\bar{x} \in [x_1, x_2]$ ,  $\lambda e_1 + (1-\lambda)e_2 \in R^{-1}(\bar{x})$  . By quasiconvexity of  $f$ ,  $f(\bar{x}) \leq \max\{f(x_1), f(x_2)\} < c$  and since  $\bar{x} \in R(\lambda e_1 + (1-\lambda)e_2)$ , also  $f^*(\lambda e_1 + (1-\lambda)e_2) < c$ , i.e.,  $\lambda e_1 + (1-\lambda)e_2 \in L_c^0$  .

An essentially equivalent result was proved recently by Oettli (1981). He considers constraints of the form  $e \in R'(x)$  and calls  $R'$  quasiconvex if  $R'(M)$  is convex for any convex set  $M \subset E^n$ .

Consistent with the definitions given in Section 2 we call the map  $R: E^k \rightarrow E^n$  "closure quasiconvex" on a convex set  $S \subset E^k$  if the map  $clR$  is quasiconvex on  $S$ , and call  $R$  "hull quasiconvex" on  $S$  if the map  $convR$  is quasiconvex on  $S$ . We use these notions to extend Proposition 3.8 in the following two results. Their proofs are not given, since they easily follow from Proposition 3.8 and Remarks 3.5 and 3.7, respectively.

Proposition 3.9. Consider the parametric optimization problem  $P'(e)$ . If  $f$  is quasiconvex and upper semicontinuous on the set  $cl(R(S))$ ,  $R$  is closure quasiconvex on  $S$  and  $S$  is convex, then  $f^*$  is quasiconvex on  $S$ .

Proposition 3.10. Consider the parametric optimization problem  $P'(e)$ . If  $f$  is quasimonotonic on the set  $conv(R(S))$ ,  $R$  is hull quasiconvex on  $S$  and  $S$  is convex, then  $f^*$  is quasiconvex on  $S$ .

A function  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  is called strictly quasiconvex on a convex set  $M \subset E^r$  (Pol'yak (1966), Ponstein (1967), who calls it unnamed convex, and Ortega and Rheinboldt (1970)) if, for all  $x_1, x_2 \in M$ ,  $x_1 \neq x_2$  and  $\lambda \in (0, 1)$ ,

$$\phi(\lambda x_1 + (1-\lambda)x_2) < \max \{\phi(x_1), \phi(x_2)\}.$$

We note that some authors reserve the name strict quasiconvexity for another property of a function (see the definition following Proposition 3.14). Our terminology is consistent with that in Avriel et. al. (1981). Note that a strictly quasiconvex function is also quasiconvex.

Proposition 3.11. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly strictly quasiconvex on the set  $\{(x, e) | x \in R(e), e \in S\}$ ,  $R$  is essentially convex or convex on  $S$ ,  $S$  is convex and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is strictly quasiconvex on  $S$ .

Proof

Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$ ,  $\lambda \in (0,1)$  and denote  $e_\lambda = \lambda e_1 + (1-\lambda)e_2$ .

By our assumptions there exist  $x_1^* \in S^*(e_1)$  and  $x_2^* \in S^*(e_2)$ , so

by strict quasiconvexity of  $f$  and convexity of  $R$  we obtain

$$\max \{f^*(e_1), f^*(e_2)\} = \max \{f(x_1^*, e_1), f(x_2^*, e_2)\}$$

$$> f(\lambda x_1^* + (1-\lambda)x_2^*, e_\lambda) \geq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} f(\lambda x_1 + (1-\lambda)x_2, e_\lambda)$$

$$\geq \inf_{x \in R(e_\lambda)} f(x, e_\lambda) = f^*(e_\lambda)$$

i.e.,  $f^*$  is strictly quasiconvex on  $S$ .

A specialization of this result to the parametric NLP problem  $P_3(e)$  can be obtained using Proposition 2.1. For the problem  $P'(e)$  we obtain another result (note that Proposition 3.11 cannot be applied in this case).

Proposition 3.12. Consider the parametric optimization problem  $P'(e)$ .

If  $f$  is strictly quasiconvex on the set  $R(S)$ ,  $R$  is convex on  $S$ ,  $S$  is convex,  $S^*(e) \neq \emptyset$  for all  $e \in S$  and  $S^*(e_1) \neq S^*(e_2)$  if  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$ , then  $f^*$  is strictly quasiconvex on  $S$ .

Proof

Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$ ,  $\lambda \in (0,1)$  and denote  $e_\lambda = \lambda e_1 + (1-\lambda)e_2$ .

By our assumptions there exist  $x_1^* \in S^*(e_1)$ ,  $x_2^* \in S^*(e_2)$ ,

$x_1^* \neq x_2^*$  and

$$\max \{f^*(e_1), f^*(e_2)\} = \max \{f(x_1^*), f(x_2^*)\}$$

$$> f(\lambda x_1^* + (1-\lambda)x_2^*) \geq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} f(\lambda x_1 + (1-\lambda)x_2) \geq \inf_{x \in R(e_\lambda)} f(x) = f^*(e_\lambda)$$

i.e.,  $f^*$  is strictly quasiconvex on  $S$ .

Remark 3.13. Note that the sets  $S^*(e)$  are actually singletons in this result, since  $f$  is strictly quasiconvex and  $R$  is convex.

In order to strengthen the last result we introduce the following notion. We call the map  $R: E^k \rightarrow E^n$  "strictly quasiconvex" on a convex set  $S \subset E^k$  if, for any  $x_1, x_2 \in E^n$ ,  $e_1 \in R^{-1}(x_1) \cap S$ ,  $e_2 \in R^{-1}(x_2) \cap S$ ,  $e_1 \neq e_2$  and  $\lambda \in (0,1)$ ,  $\lambda e_1 + (1-\lambda)e_2 \in R^{-1}((x_1, x_2)) \cap S$ . It is clear that (essential) convexity of  $R$  implies strict quasiconvexity of  $R$ , which in turn implies quasiconvexity of  $R$ .

Proposition 3.14. Consider the parametric optimization problem  $P'(e)$ . If  $f$  is strictly quasiconvex on the set  $R(S)$ ,  $R$  is strictly quasiconvex on  $S$ ,  $S$  is convex,  $S^*(e) \neq \emptyset$  for all  $e \in S$  and  $S^*(e_1) \neq S^*(e_2)$  if  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$ , then  $f^*$  is strictly quasiconvex on  $S$ .

Proof

Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$  and  $\lambda \in (0,1)$ . Denote  $e_\lambda = \lambda e_1 + (1-\lambda)e_2$ .

By the assumptions there exist  $x_1^* \in S^*(e_1)$  and  $x_2^* \in S^*(e_2)$ ,

$x_1^* \neq x_2^*$ . Thus,  $x_k^* \in R(e_k)$ ,  $k=1,2$ , so  $e_k \in R^{-1}(x_k^*) \cap S$ ,  $k=1,2$

and by strict quasiconvexity of  $R$ ,  $e_\lambda \in R^{-1}((x_1^*, x_2^*)) \cap S$ .

This implies that for some  $\bar{x} \in (x_1^*, x_2^*)$ ,  $\bar{x} \in R(e_\lambda)$ .

Therefore

$$\max \{f^*(e_1), f^*(e_2)\} = \max \{f(x_1^*), f(x_2^*)\} > f(\bar{x})$$

$$\geq \inf_{x \in R(e_\lambda)} f(x) = f^*(e_\lambda)$$

$$x \in R(e_\lambda),$$

i.e.,  $f^*$  is strictly quasiconvex on  $S$ .

A function  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  is called semistrictly quasi-convex on a convex set  $M \subset E^r$  (Avriel et. al. (1981); many authors call this property strict quasiconvexity, see e.g., Mangasarian (1965, 1969) and Ponstein (1967)) if, for all  $x_1, x_2 \in M$  and  $\lambda \in (0,1)$ ,

$$\phi(x_1) < \phi(x_2) \text{ implies that } \phi(\lambda x_1 + (1-\lambda)x_2) < \phi(x_2) .$$

Note that if  $\phi$  is strictly quasiconvex, then it is semistrictly quasi-convex. However, in general, a semistrictly quasiconvex function is also quasiconvex only if it is continuous (Avriel et. al. (1981)).

Proposition 3.15. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly semistrictly quasiconvex on the set  $\{(x,e) | x \in R(e), e \in S\}$ ,  $R$  is essentially convex or convex on  $S$ ,  $S$  is convex and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is semistrictly quasiconvex on  $S$ .

Proof

Let  $e_1, e_2 \in S$ ,  $\lambda \in (0,1)$  and denote  $e_\lambda = \lambda e_1 + (1-\lambda)e_2$ .

Suppose that  $f^*(e_1) < f^*(e_2)$  (so that  $e_1 \neq e_2$ ).

By our assumptions there exist  $x_1^* \in S^*(e_1)$  and  $x_2^* \in S^*(e_2)$ , so by semistrict quasiconvexity of  $f$  and convexity of  $R$  we obtain

$$\begin{aligned}
f^*(e_2) &= f(x_2^*, e_2) > f(\lambda x_1^* + (1-\lambda)x_2^*, e_2) \\
&\geq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} f(\lambda x_1 + (1-\lambda)x_2, e_\lambda) \geq \inf_{x \in R(e_\lambda)} f(x, e_\lambda) = f^*(e_\lambda),
\end{aligned}$$

i.e.,  $f^*$  is semistrictly quasiconvex on  $S$ .

This result can be readily specialized to the parametric NLP problem  $P_3(e)$  using Proposition 2.1. Note that unlike the situation for a strictly quasiconvex function, if  $f(x)$  is semistrictly quasiconvex, then  $f(x, e) = f(x)$  is jointly semistrictly quasiconvex. Thus, Proposition 3.15 is applicable to the problem  $P'(e)$  as well. We state this result without proof.

Corollary 3.16. Consider the parametric optimization problem  $P'(e)$ . If  $f$  is semistrictly quasiconvex on the set  $R(S)$ ,  $R$  is convex on  $S$ ,  $S$  is convex and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is semistrictly quasiconvex on  $S$ .

This result can be strengthened using the notion of strict quasiconvexity for maps.

Proposition 3.17. Consider the parametric optimization problem  $P'(e)$ . If  $f$  is semistrictly quasiconvex on the set  $R(S)$ ,  $R$  is strictly quasiconvex on  $S$ ,  $S$  is convex and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is semistrictly quasiconvex on  $S$ .

Proof

Let  $e_1, e_2 \in S$ ,  $\lambda \in (0, 1)$  and denote  $e_\lambda = \lambda e_1 + (1-\lambda)e_2$ . Suppose that  $f^*(e_1) < f^*(e_2)$  (this implies that  $e_1 \neq e_2$ ). Since  $S^*(e) \neq \emptyset$ ,  $e \in S$ , there exist  $x_1^* \in S^*(e_1)$  and  $x_2^* \in S^*(e_2)$ , hence  $x_k^* \in R(e_k)$ ,  $k=1, 2$  and  $e_k \in R^{-1}(x_k^*) \cap S$ ,  $k=1, 2$ . By strict quasiconvexity of  $R$ ,  $e_\lambda \in R^{-1}((x_1^*, x_2^*)) \cap S$ , i.e., for some  $\bar{x} \in (x_1^*, x_2^*)$ ,  $\bar{x} \in R(e_\lambda)$ .

Thus, by semistrict quasiconvexity of  $f$ ,

$$f^*(e_2) = f(x_2^*) > f(\bar{x}) \geq \inf_{x \in R(e_\lambda)} f(x) = f^*(e_\lambda),$$

i.e.,  $f^*$  is semistrictly quasiconvex on  $S$ .

Analogous to the definition of uniform convexity (see Ortega and Rheinboldt (1970)) we define "uniform quasiconvexity." A function  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  is called "uniformly quasiconvex with  $d(\cdot)$ " on a convex set  $McE^r$  (Pol'yak (1966), see also Ortega and Rheinboldt (1970) for a slightly different definition) if, for all  $x_1, x_2 \in M$  and  $\lambda \in (0,1)$ ,

$$\phi(\lambda x_1 + (1-\lambda)x_2) \leq \max \{\phi(x_1), \phi(x_2)\} - \lambda(1-\lambda)d(\|x_1 - x_2\|),$$

where  $d: [0, \infty) \rightarrow [0, \infty)$  is an increasing function, with

$d(t) > 0$  for  $t > 0$  and  $d(0) = 0$ , and  $\|\cdot\|$  is an arbitrary norm in  $E^r$ .

**Proposition 3.18.** Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly uniformly quasiconvex with  $d(\cdot)$  on the set  $\{(x, e) \mid x \in R(e), e \in S\}$ ,  $R$  is essentially convex or convex on  $S$ , and  $S$  is a convex set, then  $f^*$  is uniformly quasiconvex with  $d(\cdot)$  on  $S$ .

#### Proof

Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$  and  $\lambda \in (0,1)$ . By our assumptions and the properties of  $d$  and the norm we obtain

$$\begin{aligned} f^*(\lambda e_1 + (1-\lambda)e_2) &= \inf_{x \in R(\lambda e_1 + (1-\lambda)e_2)} f(x, \lambda e_1 + (1-\lambda)e_2) \leq \\ &\leq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} f(\lambda x_1 + (1-\lambda)x_2, \lambda e_1 + (1-\lambda)e_2) \end{aligned}$$

$$\begin{aligned}
&\leq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} [\max \{f(x_1, e_1), f(x_2, e_2)\} - \lambda(1-\lambda) d(\|(x_1, e_1) - (x_2, e_2)\|)] \\
&\leq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} [\max \{f(x_1, e_1), f(x_2, e_2)\} - \lambda(1-\lambda) d(\|e_1 - e_2\|)] \\
&= \max_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} \{ \inf f(x_1, e_1), \inf f(x_2, e_2) \} - \lambda(1-\lambda) d(\|e_1 - e_2\|) \\
&= \max \{f^*(e_1), f^*(e_2)\} - \lambda(1-\lambda) d(\|e_1 - e_2\|), \\
&\text{i.e., } f^* \text{ is uniformly quasiconvex with } d(\cdot) \text{ on } S.
\end{aligned}$$

Remark 3.19. Note that the above result would still hold if we only assumed that, for any  $x_1, x_2 \in M$  and  $\lambda \in (0,1)$ ,

$$f(\lambda x_1 + (1-\lambda)x_2, \lambda e_1 + (1-\lambda)e_2) \leq \max \{f(x_1, e_1), f(x_2, e_2)\} - \lambda(1-\lambda) d(\|e_1 - e_2\|).$$

This property combines uniform quasiconvexity of  $f$  in  $e$  with joint quasiconvexity of  $f$  in  $(x, e)$ .

Observe that Proposition 3.18 can be specialized to the NLP problems  $P_3(e)$  and  $P_2(e)$  using Proposition 2.1 and 2.2, respectively.

A function  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  is called "homogeneous quasiconvex" ("homogeneous quasiconcave") on a convex cone  $K \subset E^r$  if it is both positively homogeneous and quasiconvex (quasiconcave) on  $K$ . Using the notion of homogeneous convexity of  $R$  (see Section 2) we easily obtain the following result.

Proposition 3.20. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly homogeneous quasiconvex on  $E^n \times K$ ,  $R$  is homogeneous convex on  $K$  and  $K$  is a convex cone, then  $f^*$  is homogeneous quasiconvex on  $K$ .



Proof

Follows directly from Propositions 3.1 and 2.3 taken together.

Specialization of this result to the parametric NLP problem  $P_3(e)$  is given next.

Corollary 3.21. Consider the parametric NLP problem  $P_3(e)$ . If  $f$  is jointly homogeneous quasiconvex on  $M \times K$ ,  $\{g_i\}$  are jointly homogeneous quasiconcave on  $M \times K$ ,  $\{h_j\}$  are jointly linear on  $M \times K$ , and  $M$  and  $K$  are convex cones, then  $f^*$  is homogeneous convex on  $K$ .

Proof

Follows immediately from Proposition 3.20, 2.1 and 2.3.

4. Quasiconcavity and quasimonotonicity of the optimal value function

Recall from Section 2 that a function  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  is called "quasiconcave" on a convex set  $M \subset E^r$  if, for all  $x_1, x_2 \in M$  and  $\lambda \in (0,1)$ ,

$$\phi(\lambda x_1 + (1-\lambda)x_2) \geq \min \{\phi(x_1), \phi(x_2)\}.$$

Concavity of  $R$  was defined in Section 2.

Proposition 4.1. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly quasiconcave on the set  $\{(x,e) \mid x \in R(e), e \in S\}$ ,  $R$  is concave on  $S$ , and  $S$  is convex, then  $f^*$  is quasiconcave on  $S$ .

Proof

Let  $e_1, e_2 \in S$  and  $\lambda \in (0,1)$ . Using quasiconcavity of  $f$  and concavity of  $R$  we obtain

$$\begin{aligned} f^*(\lambda e_1 + (1-\lambda)e_2) &= \inf_{x \in R(\lambda e_1 + (1-\lambda)e_2)} f(x, \lambda e_1 + (1-\lambda)e_2) \geq \\ &\geq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} f(\lambda x_1 + (1-\lambda)x_2, \lambda e_1 + (1-\lambda)e_2) \geq \end{aligned}$$

$$\begin{aligned}
& \geq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} \min \{f(x_1, e_1), f(x_2, e_2)\} = \\
& = \min \left\{ \inf_{x_1 \in R(e_1)} f(x_1, e_1), \inf_{x_2 \in R(e_2)} f(x_2, e_2) \right\} = \\
& = \min \{f^*(e_1), f^*(e_2)\}, \text{ i.e., } f^* \text{ is quasiconcave on } S.
\end{aligned}$$

Remark 4.2. Note that the intermediate function in the proof of Proposition 4.1 given by

$$\begin{aligned}
\tilde{f}^*(\lambda) &= \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} f(\lambda x_1 + (1-\lambda)x_2, \lambda e_1 + (1-\lambda)e_2),
\end{aligned}$$

where  $e_1, e_2 \in S$  are fixed, is quasiconcave on  $[0,1]$  by Proposition 4.9 (since  $R(e_1), R(e_2)$  are fixed and  $f(\lambda x_1 + (1-\lambda)x_2, \lambda e_1 + (1-\lambda)e_2)$  is quasiconcave in  $\lambda$  for any fixed  $x_1, x_2$ ). Thus  $\tilde{f}^*(\lambda)$  is a quasiconcave lower bound on  $f^*$  on the interval  $[e_1, e_2]$ , which is sharper than the constant bound given by  $\min \{f^*(e_1), f^*(e_2)\}$ .

The following result strengthens Proposition 4.1.

Proposition 4.3. Consider the general parametric optimization Problem  $P(e)$ . If  $f$  is jointly quasiconcave on the set  $\{(x, e) \mid x \in \text{conv}(R(e)), e \in S\}$ ,  $R$  is hull concave on  $S$  and  $S$  is convex, then  $f^*$  is quasiconcave on  $S$ .

Proof

Define  $\tilde{R}(e) = \text{conv}(R(e))$  for  $e \in S$ . Denote  $\tilde{f}^*(e) = \inf_{x \in \tilde{R}(e)} \{f(x, e)\}$ . By Proposition 4.1,  $\tilde{f}^*$  is quasiconcave on  $S$ , since  $\tilde{R}$  is concave on  $S$ . Also, Remark 3.7 implies that  $\tilde{f}^*(e) = f^*(e)$  for  $e \in S$  and proves the result.

Another extension of Proposition 4.1 is given next.

Proposition 4.4. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly quasiconcave on the set  $\{(x,e) \mid x \in \text{cl}(R(e)), e \in S\}$  and upper semicontinuous in  $x$  on  $\text{cl}(R(S))$  for every  $e \in S$ ,  $R$  is closure concave on  $S$  and  $S$  is a convex set, then  $f^*$  is quasiconcave on  $S$ .

Proof

Denote  $\bar{R}(e) = \text{cl}(R(e))$ ,  $e \in S$  and  $\bar{f}^*(e) = \inf_x \{f(x,e) \mid x \in \bar{R}(e)\}$ .

By Proposition 4.1,  $\bar{f}^*$  is quasiconcave on  $S$ , since  $\bar{R}$  is concave on  $S$ . In view of Remark 3.5,  $\bar{f}^*(e) = f^*(e)$  for  $e \in S$ , proving the result.

Remark 4.5. Proposition 4.4 will remain true if we assume only that  $\text{conv}R$  is closure concave on  $S$ .

We call  $R: E^k \rightarrow E^n$  a "quasiconcave" point-to-set map on a convex set  $S \subset E^k$  if, for all  $e_1, e_2 \in S$  and  $\lambda \in (0,1)$ ,

$$R(\lambda e_1 + (1-\lambda)e_2) \subset \text{conv}(R(e_1) \cup R(e_2)).$$

Equivalently, the map  $R$  is called quasiconcave on a convex set  $S$  if  $R^+(M) \cap S$  (where we define  $R^+(M) = \{e \in E^k \mid R(e) \subset M\}$ , see Berge (1963)) is convex for any convex set  $M$  (both definitions coincide since they are in turn equivalent to the following condition: for any  $e_1, e_2 \in S$ ,

$\lambda \in (0,1)$  and convex set  $M$ ,  $R(e_1) \subset M$  and  $R(e_2) \subset M$  implies

$R(\lambda e_1 + (1-\lambda)e_2) \subset M$ ). Observe the analogy between this definition and

the definition of quasiconvexity for point-to-set maps given in Section 3.

We also note that  $R$  is quasiconcave if and only if  $\text{conv}R$  is quasiconcave.

Since a map (hull) concave on  $S$  is also quasiconcave on  $S$ , the next result strengthens the result of Propositions 4.1 and 4.3 for the problem  $P'(e)$ .

Proposition 4.6. Consider the parametric optimization problem  $P'(e)$ . If  $f$  is quasiconcave on the set  $R(S)$ ,  $R$  is quasiconcave on  $S$  and  $S$  is convex, then  $f^*$  is quasiconcave on  $S$ .

Proof

Let  $e_1, e_2 \in S$  and  $\lambda \in (0,1)$ . Suppose that for some  $c \in E^1 \cup \{-\infty\}$   $f^*(e_1) > c$  and  $f^*(e_2) > c$ . Then  $f(x) > c$  for all  $x \in R(e_1)$  and for all  $x \in R(e_2)$ , i.e.  $R(e_1) \cup R(e_2) \subset U_c^0 = \{x | f(x) > c\}$ . By quasiconcavity of  $f$   $U_c^0$  is convex, hence  $\text{conv}(R(e_1) \cup R(e_2)) \subset U_c^0$ . By quasiconcavity of  $R$  also  $R(\lambda e_1 + (1-\lambda)e_2) \subset U_c^0$ , i.e.,  $f(x) > c$  for all  $x \in R(\lambda e_1 + (1-\lambda)e_2)$  implying that  $f^*(\lambda e_1 + (1-\lambda)e_2) \geq c$ .

The next result gives yet another sufficient condition for quasiconcavity of  $f^*$  for the problem  $P'(e)$ , by strengthening the assumption on  $R$  and dispensing with the assumption on  $f$ .

Proposition 4.7. Consider the parametric optimization problem  $P'(e)$ . If  $f$  is arbitrary,  $S$  is convex, and if, for any  $e_1, e_2 \in S$  and  $\lambda \in (0,1)$ ,

$R(\lambda e_1 + (1-\lambda)e_2) \subset R(e_1) \cup R(e_2)$ , then  $f^*$  is quasiconcave on  $S$ .

Proof

Let  $e_1, e_2 \in S$  and  $\lambda \in (0,1)$ . By our assumption

$$f^*(\lambda e_1 + (1-\lambda)e_2) = \inf_{x \in R(\lambda e_1 + (1-\lambda)e_2)} f(x) \geq \inf_{x \in R(e_1) \cup R(e_2)} f(x) =$$

$$= \min \{ \inf_{x \in R(e_1)} f(x), \inf_{x \in R(e_2)} f(x) \} = \min \{ f^*(e_1), f^*(e_2) \},$$

i.e.  $f^*$  is quasiconcave on  $S$ .

The following result provides an application of Proposition 4.7 to duality theory in quasiconvex programming (Luenberger (1968) and Oettli (1981)).

**Proposition 4.8.** Consider the general parametric optimization problem  $P(e)$ . Define the dual function  $\sigma: E^r \rightarrow E^1$  as follows

$$\sigma(z) = \inf_{(x,e)} \{ f(x,e) \mid x \in R(e), z^T e \leq 0 \}.$$

Then, for arbitrary  $f$  and  $R$ ,  $\sigma$  is quasiconcave on  $E^r$ .

#### Proof

Define the feasible set in the above minimization by  $\tilde{R}(z) = \{(x,e) \in G(R) \mid z^T e \leq 0\}$ . We want to show that for any  $z_1, z_2 \in E^r$  and  $\lambda \in (0,1)$   $\tilde{R}(\lambda z_1 + (1-\lambda)z_2) \subset \tilde{R}(z_1) \cup \tilde{R}(z_2)$ . Let  $(x,e) \in \tilde{R}(\lambda z_1 + (1-\lambda)z_2)$ , i.e.  $(x,e) \in G(R)$  and  $(\lambda z_1 + (1-\lambda)z_2)^T e \leq 0$ . Suppose that  $(x,e) \notin \tilde{R}(z_1) \cup \tilde{R}(z_2)$ . Then  $z_1^T e > 0$  and  $z_2^T e > 0$ , since  $(x,e) \in G(R)$ . But this implies that  $(\lambda z_1 + (1-\lambda)z_2)^T e > 0$  which is a contradiction. The result follows from Proposition 4.7.

The above result was recently proved by Oettli (1981) in the context of quasiconvex programming when  $f(x,e) = f(x)$ . He also notes that the result remains true if we replace  $z^T e \leq 0$  by  $\tilde{g}(z,e) \leq 0$ , where  $\tilde{g}$  is quasiconcave in  $z$  for any  $e \in E^r$ . Also, an earlier version of this result was obtained by Luenberger (1968) for the standard rhs parametric NLP problem with no equality constraints present and  $\sigma$  defined on  $E_+^r = \{z \in E^r \mid z_i \geq 0, i=1, \dots, r\}$ . The name dual function is justified by the formula

$$\sigma(z) = \inf_e \{ f^*(e) \mid z^T e \leq 0 \}$$

and the fact that under certain conditions  $f^*(0) = \max_{z \in E^r} \sigma(z)$ .

Finally, we give the following two simple results.

Proposition 4.9. Consider the special problem  $P_1(e)$ . If  $f$  is quasiconcave in  $e$  on  $S$  for any  $x \in M$ ,  $S$  is convex and  $M$  is arbitrary, then  $f^*$  is quasiconcave on  $S$ .

Proof

Let  $e_1, e_2 \in S$  and  $\lambda \in (0,1)$ . By our assumptions

$$\begin{aligned} f^*(\lambda e_1 + (1-\lambda)e_2) &= \inf_{x \in M} f(x, \lambda e_1 + (1-\lambda)e_2) \geq \\ &\inf_{x \in M} \min \{f(x, e_1), f(x, e_2)\} = \\ &\min \left\{ \inf_{x \in M} f(x, e_1), \inf_{x \in M} f(x, e_2) \right\} = \min \{f^*(e_1), f^*(e_2)\}, \\ &\text{i.e. } f^* \text{ is quasiconcave on } S. \end{aligned}$$

Note that this result has a more general counterpart:  $\inf_{i \in I} f_i$  is quasiconcave if  $\{f_i\}$  are quasiconcave and  $I$  is an arbitrary index set. One can also show that this result is a corollary of Proposition 4.7.

Proposition 4.10. Consider the general parametric optimization Problem  $P(e)$ . If  $f$  is quasiconcave in  $x$  on  $M$  for all  $e \in S$  and is quasiconcave in  $e$  on  $S$  for all  $x \in M$ ,  $S$  is convex and  $\text{conv}(R(e)) = M$  for all  $e \in S$ , then  $f^*$  is quasiconcave on  $S$ .

Proof

Define  $\tilde{f}^*(e) = \inf_x \{f(x, e) \mid x \in M\}$  for  $e \in S$ . By Proposition 4.9  $\tilde{f}^*$  is quasiconcave on  $S$ . Applying Remark 3.7 we obtain  $\tilde{f}^*(e) = \inf_{x \in R(e)} f(x, e) = f^*(e)$  for  $e \in S$ , thus  $f^*$  is also quasiconcave on  $S$ .

We call  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  "strictly quasiconcave" on a convex set  $M \subset E^r$  if, for all  $x_1, x_2 \in M$ ,  $x_1 \neq x_2$  and  $\lambda \in (0,1)$ ,

$$\phi(\lambda x_1 + (1-\lambda)x_2) > \min\{\phi(x_1), \phi(x_2)\},$$

i.e., if  $\phi$  is strictly quasiconvex on  $M$  (see the remarks following the definition of strict quasiconvexity in Section 3).

**Proposition 4.11.** Consider the general parametric optimization Problem  $P(e)$ . If  $f$  is jointly strictly quasiconcave on the set  $\{(x,e) \mid x \in \text{conv}(R(e)), e \in S\}$ ,  $R$  is hull concave or concave on  $S$ ,  $S$  is convex and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is strictly quasiconcave on  $S$ .

#### Proof

In view of Remark 3.7 it is enough to prove this result for  $R$  concave. Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$  and  $\lambda \in (0,1)$ . Denote  $e_\lambda = \lambda e_1 + (1-\lambda)e_2$ . By our assumptions there exists  $x^* \in S^*(e_\lambda)$  and by concavity of  $R$   $x^* = \lambda x_1 + (1-\lambda)x_2$  for some  $x_1 \in R(e_1)$ ,  $x_2 \in R(e_2)$ . Using strict quasiconcavity of  $f$  we obtain

$$\begin{aligned} f^*(e_\lambda) &= f(x^*, e_\lambda) = f(\lambda x_1 + (1-\lambda)x_2, e_\lambda) \\ &> \min \{f(x_1, e_1), f(x_2, e_2)\} \\ &\geq \min \left\{ \inf_{x \in R(e_1)} f(x, e_1), \inf_{x \in R(e_2)} f(x, e_2) \right\} = \min \{f^*(e_1), f^*(e_2)\}, \end{aligned}$$

i.e.  $f^*$  is strictly quasiconcave on  $S$ .

For the problem  $P'(e)$  we obtain the following result (strict concavity and strict hull concavity for maps were introduced in Section 2).

**Proposition 4.12.** Consider the parametric optimization problem  $P'(e)$ . If  $f$  is strictly quasiconcave on the set  $\text{conv}(R(S))$ ,  $R$  is strictly hull concave or strictly concave on  $S$ ,  $S$  is convex and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is strictly quasiconcave on  $S$ .

#### Proof

By Remark 3.7 it suffices to give the proof for  $R$  strictly concave. Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$  and  $\lambda \in (0,1)$ . Denote  $e_\lambda = \lambda e_1 + (1-\lambda)e_2$ . By our assumptions there exists  $x^* \in S^*(e_\lambda)$  and by strict concavity

of  $R$   $x^* = \lambda x_1 + (1-\lambda)x_2$  for some  $x_1 \in R(e_1)$ ,  $x_2 \in R(e_2)$ ,  $x_1 \neq x_2$ .

Using strict quasiconcavity of  $f$  we have

$$\begin{aligned} f^*(e_\lambda) &= f(x^*) = f(\lambda x_1 + (1-\lambda)x_2) > \min \{f(x_1), f(x_2)\} \\ &\geq \min \{ \inf_{x \in R(e_1)} f(x), \inf_{x \in R(e_2)} f(x) \} = \min \{f^*(e_1), f^*(e_2)\}, \end{aligned}$$

i.e.  $f^*$  is strictly quasiconcave on  $S$ .

The result in Proposition 4.12 can be generalized using the notion introduced next. A point-to-set map  $R: E^k \rightarrow E^n$  will be called "strictly quasiconcave" on a convex set  $S \subset E^k$  if for  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$ ,  $\lambda \in (0,1)$  and  $x \in R(\lambda e_1 + (1-\lambda)e_2)$  there exist  $x_1, x_2 \in R(e_1) \cup R(e_2)$ ,  $x_1 \neq x_2$  and  $\mu \in (0,1)$  such that  $x = \mu x_1 + (1-\mu)x_2$ . It is easily seen that a strictly concave map on  $S$  is also strictly quasiconcave on  $S$ , which in turn is quasiconcave on  $S$ .

Proposition 4.13. Consider the parametric optimization Problem  $P'(e)$ . If  $f$  is strictly quasiconcave on the set  $R(S)$ ,  $R$  is strictly quasiconcave on  $S$ ,  $S$  is convex and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is strictly quasiconcave on  $S$ .

#### Proof

Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$  and  $\lambda \in (0,1)$ . Denote  $e_\lambda = \lambda e_1 + (1-\lambda)e_2$ .

By our assumptions there exists  $x^* \in S^*(e_\lambda)$ . Strict quasiconcavity of  $R$  implies that  $x^* = \mu x_1 + (1-\mu)x_2$  for some  $x_1, x_2 \in R(e_1) \cup R(e_2)$ ,  $\mu \in (0,1)$ . By strict quasiconcavity of  $f$  we obtain

$$\begin{aligned} f^*(e_\lambda) &= f(x^*) = f(\mu x_1 + (1-\mu)x_2) > \min \{f(x_1), f(x_2)\} \\ &\geq \inf_{x \in R(e_1) \cup R(e_2)} f(x) = \min \{ \inf_{x \in R(e_1)} f(x), \inf_{x \in R(e_2)} f(x) \} = \min \{f^*(e_1), f^*(e_2)\}, \end{aligned}$$

i.e.  $f^*$  is strictly quasiconcave on  $S$ .



In the next result we consider the Problem  $P_1(e)$ .

**Proposition 4.14.** Consider the special Problem  $P_1(e)$ . If  $f$  is strictly quasiconcave in  $e$  on  $S$  for all  $x \in M$ ,  $S$  is convex,  $M$  is arbitrary and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is strictly quasiconcave on  $S$ .

Proof

Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$ ,  $\lambda \in (0,1)$  and denote  $e_\lambda = \lambda e_1 + (1-\lambda)e_2$ .

By our assumption there exists  $x^* \in S^*(e_\lambda)$ . Using strict quasiconcavity of  $f$  we obtain

$$\begin{aligned} f^*(e_\lambda) &= f(x^*, e_\lambda) > \min \{f(x^*, e_1), f(x^*, e_2)\} \\ &\geq \min \left\{ \inf_{x \in M} f(x, e_1), \inf_{x \in M} f(x, e_2) \right\} = \min \{f^*(e_1), f^*(e_2)\}, \end{aligned}$$

i.e.  $f^*$  is strictly quasiconcave on  $S$ .

We call  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  "semistrictly quasiconcave" on a convex set  $M \subset E^r$  (see Avriel et.al. (1981)) if, for all  $x_1, x_2 \in M$  and  $\lambda \in (0,1)$ ,

$$\phi(x_2) > \phi(x_1) \text{ implies that } \phi(\lambda x_1 + (1-\lambda)x_2) > \phi(x_1),$$

i.e. if  $-\phi$  is semistrictly quasiconvex on  $M$ .

(see the remarks concerning semistrict quasiconvexity in Section 3). The next example shows that a result analogous to Propositions 4.11-4.13 does not hold in this case.

**Example 4.15.** Consider the problem  $P^1(e)$ . Let  $f: E^1 \rightarrow E^1$  and  $R: E^1 \rightarrow E^1$  be given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases} \text{ and } R(e) = \{x \mid x \leq e\}, e \in E^1.$$

Then,  $f$  is jointly semistrictly quasiconcave (but not quasiconcave) and  $R$  is concave (even affine) on  $E^1$ , but

$$f^*(e) = \begin{cases} 1, & \text{if } e < 0 \\ 0, & \text{if } e \geq 0 \end{cases} \text{ is not semistrictly quasiconcave}$$

(it is only quasiconcave) on  $E^1$ . Also,  $S^*(e) \neq \phi$  for  $e \in E^1$ .

It would be interesting to provide a similar example with  $f$  continuous so that semistrict quasiconcavity of  $f$  would imply quasiconcavity of  $f$ .

We call  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  "uniformly quasiconcave with  $d(\cdot)$ " on a convex set  $M \subset E^r$  if  $-\phi$  is uniformly quasiconvex with  $d(\cdot)$  on  $M$  (See Section 3 for the latter definition).

Proposition 4.16. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly uniformly quasiconcave with  $d(\cdot)$  on the set  $\{(x, e) \mid x \in \text{conv}(R(e)), e \in S\}$ ,  $R$  is hull concave or concave on  $S$  and  $S \subset E^k$  is a convex set, then  $f^*$  is uniformly quasiconcave with  $d(\cdot)$  on  $S$ .

#### Proof

In view of Remark 3.7 it suffices to prove the result in case of  $R$  concave. Let  $e_1, e_2 \in S$  and  $\lambda \in (0, 1)$ . By our assumptions and the properties of  $d$  and the norm we obtain

$$\begin{aligned} f^*(\lambda e_1 + (1-\lambda)e_2) &= \inf_{x \in R(\lambda e_1 + (1-\lambda)e_2)} f(x, \lambda e_1 + (1-\lambda)e_2) \\ &\geq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} f(\lambda x_1 + (1-\lambda)x_2, \lambda e_1 + (1-\lambda)e_2) \geq \\ &\geq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} [\min \{f(x_1, e_1), f(x_2, e_2)\} + \lambda(1-\lambda) d(\|(x_1, e_1) - (x_2, e_2)\|)] \end{aligned}$$

$$\begin{aligned}
& \geq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} [\min \{f(x_1, e_1), f(x_2, e_2)\} + \lambda(1-\lambda) d(|e_1 - e_2|)] \\
& = \min_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} \{\inf f(x_1, e_1), \inf f(x_2, e_2)\} + \lambda(1-\lambda) d(|e_1 - e_2|) \\
& = \min \{f^*(e_1), f^*(e_2)\} + \lambda(1-\lambda) d(|e_1 - e_2|), \\
& \text{i.e. } f^* \text{ is uniformly quasiconcave with } d(\cdot) \text{ on } S.
\end{aligned}$$

An observation analogous to Remark 3.19 holds also for Proposition 4.16.

Proposition 4.17. Consider the special problem  $P_1(e)$ . If  $f$  is uniformly quasiconcave in  $e$  with  $d(\cdot)$  on  $S$  for any  $x \in M$ ,  $S$  is convex and  $M$  is arbitrary, then  $f^*$  is uniformly quasiconcave with  $d(\cdot)$  on  $S$ .

Proof

Let  $e_1, e_2 \in S$  and  $\lambda \in (0,1)$ . By our assumptions

$$\begin{aligned}
f^*(\lambda e_1 + (1-\lambda)e_2) &= \inf_{x \in M} f(x, \lambda e_1 + (1-\lambda)e_2) \geq \\
&\geq \inf_{x \in M} [\min \{f(x, e_1), f(x, e_2)\} + \lambda(1-\lambda) d(|e_1 - e_2|)] \\
&= \min_{\substack{x \in M \\ x \in M}} \{\inf f(x, e_1), \inf f(x, e_2)\} + \lambda(1-\lambda) d(|e_1 - e_2|) \\
&= \min \{f^*(e_1), f^*(e_2)\} + \lambda(1-\lambda) d(|e_1 - e_2|), \\
&\text{i.e. } f^*(e) \text{ is uniformly quasiconcave with } d(\cdot) \text{ on } S.
\end{aligned}$$

Using the notion of homogeneous concavity for maps (see Section 2) we obtain the following counterpart of Proposition 3.20 (recalling that a homogeneous concave function was defined in Section 3).

Proposition 4.18. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly homogeneous quasiconcave on  $E^n \times K$ ,  $R$  is

homogeneous concave on  $K$ , and  $K$  is a convex cone, then  $f^*$  is homogeneous quasiconcave on  $K$ .

Proof

Follows easily from Propositions 4.1 and 2.3 taken together.

Remark 4.19. Proposition 4.18 will remain true if the map  $\text{conv}R$  is homogeneous concave on  $K$ , in view of Remark 3.7.

The next result is obtained for the problem  $P_1(e)$ .

Proposition 4.20. Consider the special problem  $P_1(e)$ . If  $f$  is homogeneous quasiconcave in  $e$  on  $K$  for all  $x \in M$ ,  $K$  is a convex cone and  $M$  is arbitrary, then  $f^*$  is homogeneous quasiconcave on  $K$ .

Proof

Follows easily from Propositions 4.9 and 2.5.

We shall state now several results providing sufficient conditions for quasimonotonicity of  $f^*$ . Recall from Section 2 that  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  is called "quasimonotonic" on a convex set  $M \subset E^r$  if, for all  $x_1, x_2 \in M$  and  $\lambda \in (0, 1)$ ,

$$\min \{ \phi(x_1), \phi(x_2) \} \leq \phi(\lambda x_1 + (1-\lambda)x_2) \leq \max \{ \phi(x_1), \phi(x_2) \}.$$

The notions of an (essentially) affine and an (essentially) hull affine point-to-set map were introduced in Section 2.

Proposition 4.21. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly quasimonotonic on the set  $\{(x, e) \mid x \in \text{conv}(R(e)), e \in S\}$ ,  $R$  is (essentially) hull affine or (essentially) affine on  $S$  and  $S$  is convex, then  $f^*$  is quasimonotonic on  $S$ .

Proof

Follows from Propositions 3.6 and 4.3 and the fact that an (essentially) affine map is also (essentially) hull affine.

The following result slightly extends Proposition 4.21.

Proposition 4.22. Consider again the general parametric optimization problem  $P(e)$ . If  $f$  is jointly quasimonotonic on the set  $\{(x,e) \mid x \in \text{cl conv}(R(e)), e \in S\}$  and upper semicontinuous in  $x$  on the set  $\text{cl conv}(R(S))$  for every  $e \in S$ ,  $\text{cl } R$  is (essentially) hull affine or (essentially) affine on  $S$  and  $S$  is convex, then  $f^*$  is quasimonotonic on  $S$ .

Proof

Follows immediately from Proposition 4.21 and Remark 3.5.

In order to strengthen Proposition 4.21 for the problem  $P'(e)$ , we introduce the following definitions. A point-to-set map  $R: E^k \rightarrow E^n$  is called "quasimonotonic" on a convex set  $S \subset E^k$  if  $R$  is both quasiconvex and quasiconcave on  $S$ , i.e. if for any convex set  $M \subset E^n$  both  $R^{-1}(M) \cap S$  and  $R^+(M) \cap S$  are convex. Further,  $R$  is called "hull quasimonotonic" on  $S$  if  $\text{conv } R$  is quasimonotonic on  $S$ . Note that hull quasimonotonicity of  $R$  on  $S$  is equivalent to hull quasiconvexity together with quasiconcavity of  $R$  on  $S$ .

Proposition 4.23. Consider the parametric optimization problem  $P'(e)$ . If  $f$  is quasimonotonic on the set  $\text{conv}(R(S))$ ,  $R$  is hull quasimonotonic or quasimonotonic on  $S$  and  $S$  is convex, then  $f^*$  is quasimonotonic on  $S$ .

Proof

The result follows from Propositions 3.8, 3.10 and 4.6 combined.

Before stating the next results we introduce the following definitions. A function  $\phi: E^r \rightarrow E^1 \cup \{-\infty, \infty\}$  is called "strictly quasimonotonic" on a convex set  $M \subset E^r$  if it is both strictly quasiconvex and strictly quasiconcave, i.e., if, for all  $x_1, x_2 \in M$ ,  $x_1 \neq x_2$  and  $\lambda \in (0,1)$ ,

$$\min \{ \phi(x_1), \phi(x_2) \} < \phi(\lambda x_1 + (1-\lambda)x_2) < \max \{ \phi(x_1), \phi(x_2) \}.$$

The point-to-set map  $R: E^k \rightarrow E^n$  is called "strictly affine" ("strictly hull affine") on a convex set  $S \subset E^k$  if  $R$  ( $\text{conv}R$ ) is both convex and strictly concave on  $S$ . A map strictly affine on  $S$  is affine on  $S$  but not conversely. We call  $R$  "strictly quasimonotonic" on  $S$  if it is both strictly quasiconvex and strictly quasiconcave on  $S$ . Strict affinity of  $R$  on  $S$  implies strict quasimonotonicity of  $R$  on  $S$ , which in turn implies quasimonotonicity of  $R$  on  $S$ .

Proposition 4.24. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly strictly quasimonotonic on the set  $\{(x,e) \mid x \in \text{conv}(R(e)), e \in S\}$ ,  $R$  is hull affine or affine on  $S$ ,  $S$  is convex and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is strictly quasimonotonic on  $S$ .

Proof

Follows from Propositions 3.11 and 4.11 and Remark 3.7.

Proposition 4.25. Consider the parametric optimization problem  $P'(e)$ . If  $f$  is strictly quasimonotonic on the set  $\text{conv}(R(S))$ ,  $R$  is strictly hull affine or strictly affine on  $S$ ,  $S$  is convex,  $S^*(e) \neq \emptyset$  for all  $e \in S$  and  $S^*(e_1) \neq S^*(e_2)$  if  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$ , then  $f^*$  is strictly quasimonotonic on  $S$ .

Proof

Follows from Propositions 3.12 and 4.12 and Remark 3.7.

The next result strengthens Proposition 4.25.

Proposition 4.26. Consider again the parametric optimization problem  $P'(e)$ . If  $f$  is strictly quasimonotonic on the set  $R(S)$ ,  $R$  is strictly quasimonotonic on  $S$ ,  $S$  is convex,  $S^*(e) \neq \emptyset$  for all  $e \in S$  and  $S^*(e_1) \neq S^*(e_2)$  if  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$ , then  $f^*$  is strictly quasimonotonic on  $S$ .

Proof

Follows from Propositions 3.14 and 4.13.

5. Pseudoconvexity and pseudoconcavity of the optimal value function

We shall consider the notion of pseudoconvexity first. A function  $\phi: E^r \rightarrow E^1$  is called "pseudoconvex" on a convex set  $M \subset E^r$  (Ortega and Rheinboldt (1970), Thompson and Parke (1973)) if, for all  $x_1, x_2 \in M$  and  $\lambda \in (0,1)$ ,

$$\phi(x_1) < \phi(x_2) \text{ implies that } \phi(\lambda x_1 + (1-\lambda)x_2) \leq \phi(x_2) - \lambda\beta(x_1, x_2),$$

where  $\beta(x_1, x_2) > 0$  depends only on  $x_1$  and  $x_2$ . This definition is an extension of the standard definition of a pseudoconvex function (Mangasarian (1965, 1969)), as shown in the next result.

Proposition 5.1. (Ortega and Rheinboldt (1970), Thompson and Parke (1973)).

Let  $\phi: E^r \rightarrow E^1$  be a differentiable function defined on an open convex set  $M \subset E^r$ . Then  $\phi$  is pseudoconvex if and only if, for all  $x_1, x_2 \in M$ ,

$$\nabla\phi(x_1)^T (x_2 - x_1) \geq 0 \text{ implies that } \phi(x_2) \geq \phi(x_1).$$

Here, we are interested in general results and hence prefer the more general definition. It is also noted that in general  $f^*$  is not differentiable, even if  $f$  is differentiable.

Proposition 5.2. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly pseudoconvex on the set  $\{(x, e) \mid x \in R(e), e \in S\}$ ,  $R$  is essentially convex or convex on  $S$ ,  $S$  is convex and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is pseudoconvex on  $S$ .

Proof

Let  $e_1, e_2 \in S$  and suppose that  $f^*(e_1) < f^*(e_2)$ . Then, since  $S^*(e) \neq \emptyset$  for all  $e \in S$ ,  $f^*(e_1) = f(x_1^*, e_1)$  for some  $x_1^* \in R(e_1)$

and  $f^*(e_2) = f(x_2^*, e_2)$  for some  $x_2^* \in R(e_2)$ . By pseudoconvexity of  $f$ ,  $f(x_1^*, e_1) < f(x_2^*, e_2)$  implies that for all  $\lambda \in (0,1)$  and some  $\beta > 0$  which depends only on  $(x_1^*, e_1)$  and  $(x_2^*, e_2)$

$$f(\lambda x_1^* + (1-\lambda)x_2^*, \lambda e_1 + (1-\lambda)e_2) \leq f(x_2^*, e_2) - \lambda\beta.$$

By convexity of  $R$ , for any  $\lambda \in (0,1)$

$$\lambda x_1^* + (1-\lambda)x_2^* \in R(\lambda e_1 + (1-\lambda)e_2), \text{ so for all } \lambda \in (0,1)$$

$$f^*(\lambda e_1 + (1-\lambda)e_2) \leq f(\lambda x_1^* + (1-\lambda)x_2^*, \lambda e_1 + (1-\lambda)e_2)$$

$$\leq f(x_2^*, e_2) - \lambda\beta = f^*(e_2) - \lambda\beta.$$

Since  $\beta$  now depends only on  $e_1$  and  $e_2$  the results is proved.

Note that Proposition 5.2 is directly applicable to  $P'(e)$  as well. It is not known whether the assumption that  $S^*(e) \neq \emptyset$  for all  $e \in S$  can be relaxed in the above result. Proposition 5.2 can be specialized to the case of the parametric NLP problem  $P_3(e)$  by assuming, as in Proposition 2.1, joint quasiconcavity or concavity of  $\{g_i\}$  and joint quasimonotonicity or affineness of  $\{h_j\}$  on  $M \times S$ , with  $M$  convex. Furthermore, Proposition 5.2 can be specialized to  $P_2(e)$  precisely as in Proposition 2.2. Finally, if  $f^*$  is known or assumed to be differentiable and  $S$  is open and convex, we can use the definition of Mangasarian (1965, 1969).

A function  $\phi: E^r \rightarrow E^1$  is called "strictly pseudoconvex" on a convex set  $M \subset E^r$  (Ortega and Rheinboldt (1970), Thompson and Parke (1973)) if, for all  $x_1, x_2 \in M$ ,  $x_1 \neq x_2$  and  $\lambda \in (0,1)$ ,

$$\phi(x_1) \leq \phi(x_2) \text{ implies that } \phi(\lambda x_1 + (1-\lambda)x_2) \leq \phi(x_2) - \lambda\beta(x_1, x_2),$$

where  $\beta(x_1, x_2) > 0$  depends only on  $x_1$  and  $x_2$ . This definition also has a differentiable counterpart (Ponstein (1967)), analogous to the definition of a pseudoconvex function (see Proposition 5.1).



Proposition 5.3. Consider the general parametric optimization problem  $P(e)$ . If  $f$  is jointly strictly pseudoconvex on the set  $\{(x,e) \mid x \in R(e), e \in S\}$ ,  $R$  is convex on  $S$ ,  $S$  is convex and  $S^*(e) \neq \emptyset$  for all  $e \in S$ , then  $f^*$  is strictly pseudoconvex on  $S$ .

Proof

Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$  and suppose that  $f^*(e_1) \leq f^*(e_2)$ . Since  $S^*(e) \neq \emptyset$  for all  $e \in S$ ,  $f(x_1^*, e_1) = f^*(e_1)$  for some  $x_1^* \in R(e_1)$  and  $f(x_2^*, e_2) = f^*(e_2)$  for some  $x_2^* \in R(e_2)$ . By strict pseudoconvexity of  $f$  there is  $\beta > 0$  depending only on  $(x_1^*, e_1)$  and  $(x_2^*, e_2)$  such that for all  $\lambda \in (0,1)$

$$f(\lambda x_1^* + (1-\lambda)x_2^*, \lambda e_1 + (1-\lambda)e_2) \leq f(x_2^*, e_2) - \lambda\beta.$$

By convexity of  $R$   $\lambda x_1^* + (1-\lambda)x_2^* \in R(\lambda e_1 + (1-\lambda)e_2)$  for all  $\lambda \in (0,1)$  so that for any  $\lambda \in (0,1)$

$$f^*(\lambda e_1 + (1-\lambda)e_2) \leq f(\lambda x_1^* + (1-\lambda)x_2^*, \lambda e_1 + (1-\lambda)e_2)$$

$$\leq f(x_2^*, e_2) - \lambda\beta = f^*(e_2) - \lambda\beta.$$

Since  $\beta$  depends only on  $e_1$  and  $e_2$  the proof is completed.

For the problem  $P'(e)$  we state another result, since Proposition 5.3 is not applicable in this case (i.e., with  $f(x,e) = f(x)$ ).

Proposition 5.4. Consider the parametric optimization problem  $P'(e)$ . If  $f$  is strictly pseudoconvex on the set  $R(S)$ ,  $R$  is convex on  $S$ ,  $S$  is convex,  $S^*(e) \neq \emptyset$  for all  $e \in S$  and  $S^*(e_1) \neq S^*(e_2)$  if  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$ , then  $f^*$  is strictly pseudoconvex on  $S$ .

Proof

Let  $e_1, e_2 \in S$ ,  $e_1 \neq e_2$  and denote  $e_\lambda = \lambda e_1 + (1-\lambda)e_2$ .

If  $f^*(e_1) \leq f^*(e_2)$ , then  $f(x_1^*) \leq f(x_2^*)$  for some  $x_1^* \in R(e_1)$ ,  $x_2^* \in R(e_2)$  and  $x_1^* \neq x_2^*$ , by our assumptions. Strict pseudoconvexity of  $f$  and convexity of  $R$  imply that for all  $\lambda \in (0,1)$  and some  $\beta > 0$  which does not depend on  $\lambda$ ,

$$\begin{aligned} f^*(e_\lambda) &= \inf_{x \in R(e_\lambda)} f(x) \leq \inf_{\substack{x_1 \in R(e_1) \\ x_2 \in R(e_2)}} f(\lambda x_1 + (1-\lambda)x_2) \leq f(\lambda x_1^* + (1-\lambda)x_2^*) \\ &\leq f(x_2^*) - \lambda\beta = f^*(e_2) - \lambda\beta, \text{ what completes the proof.} \end{aligned}$$

Remark 5.5. Note that the sets  $S^*(e)$  are actually singletons here, since strict pseudoconvexity implies strict quasiconvexity (Poinstein (1967)).

Next, we consider the notion of pseudoconcavity. A function  $\phi: E^r \rightarrow E^1$  is called "pseudoconcave" on a convex set  $M \subset E^r$  (Ortega and Rheinboldt (1970)) if  $-\phi$  is pseudoconvex on  $M$ , i.e., if, for all  $x_1, x_2 \in M$  and  $\lambda \in (0,1)$ ,

$$f(x_1) < f(x_2) \text{ implies that } f((1-\lambda)x_1 + \lambda x_2) \geq f(x_1) + \lambda\beta(x_1, x_2),$$

where  $\beta(x_1, x_2) > 0$  depends only on  $x_1$  and  $x_2$ . As with a pseudoconvex function, the above definition is an extension of the standard one (Mangasarian (1965, 1969)) and a result analogous to Proposition 5.1 holds. Unlike a strict parallel of the pseudoconvex result, however, pseudoconcavity of  $f^*$  does not seem to follow from pseudoconcavity of  $f$  and concavity of  $R$ . This is seen from Example 4.15, where the objective function is clearly pseudoconcave but  $f^*$  is not pseudoconcave. As was mentioned before,  $f$  is not quasiconcave in this example. Therefore we give one additional example with  $f$  pseudoconcave and continuous (and thus also quasiconcave).

Example 5.6. Consider the problem  $P'(e)$ . Let  $f: E^1 \rightarrow E^1$  and  $R: E^1 \rightarrow E^1$  be defined as follows:  $f(x) = e^x$ ,  $x \in E^1$  and

$$R(e) = \begin{cases} (-\infty, 0) & , \text{ if } e \leq 0 \\ \{0\} & , \text{ if } e > 0 . \end{cases}$$

It can be easily verified that  $f$  is pseudoconcave (and quasiconcave) and  $R$  is concave on  $E^1$ . However,

$$f^*(e) = \begin{cases} 0 & , e \leq 0 \\ 1 & , e > 0 \end{cases} \quad \text{is not pseudoconcave}$$

(though it is quasiconcave) on  $E^1$ .

The next example shows that  $\inf_{i \in I} f_i$  is not pseudoconcave

in general, even if  $\{f_i\}$  are pseudoconcave.

Example 5.7. Consider the problem  $P_1(e)$ . Let

$$f(x, e) = \begin{cases} xe & , \text{ if } e \leq \frac{1}{2} , x \in E^1 \\ \frac{x}{2} + e - \frac{1}{2} & , \text{ if } e > \frac{1}{2} , x \in E^1 \end{cases}$$

and  $S = [0, 1]$ ,  $M = (0, 1)$ . Then  $f$  is pseudoconcave (and quasiconcave) in  $e$  on  $S$  for all  $x \in M$ .

However,

$$f^*(e) = \begin{cases} 0 & , 0 \leq e \leq \frac{1}{2} \\ e - \frac{1}{2} & , \frac{1}{2} < e \leq 1 \end{cases} \quad \text{is not}$$

pseudoconcave (it is only quasiconcave) on  $S$ .

Note that in these two examples, the minimum was not attained for the problems of interest. This leaves open the possibility that pseudoconcavity of  $f^*$  might be proved for  $f$  both pseudoconcave and quasiconcave and  $R$  concave (or affine) under the assumption of nonempty solution sets  $S^*(e)$ , as in the pseudoconvex case (see Proposition 5.2).

A function  $\phi : E^r \rightarrow E^1$  is called "strictly pseudoconcave" on a convex set  $M \subset E^r$  if  $-\phi$  is strictly pseudoconvex on  $M$ . The following example shows that  $\inf_{i \in I} f_i$  is not always strictly pseudoconcave, even if  $\{f_i\}$  are strictly pseudoconcave.

Example 5.8. Consider the problem  $P_1(e)$ . Let  $f(x,e) = x(e^2 + e)$  for  $x \in M = (0,1)$ ,  $e \in S = (0,1)$ . Then,  $f$  is strictly pseudoconcave in  $e$  on  $S$  for all  $x \in M$ , but  $f^*(e) = 0$ ,  $e \in S$ , is not strictly pseudoconcave.

## 6. Concluding remarks

A short overview of the developments from preceding sections is given in this section. Section 2 contains the material taken from the previous paper by the authors. This material is extensively used here. In Section 3 quasiconvexity properties of the optimal value function  $f^*$  are considered. Except for Proposition 3.8, all of the results seem to be new. The basic sufficient conditions for quasiconvexity of  $f^*$  are given in Proposition 3.1. Specializations to problems  $P_3(e)$  and  $P_2(e)$  are obtained in Propositions 3.2 and 3.3. Some extensions are also given (Proposition 3.4 and 3.6). Using the notion of a quasiconvex map we strengthen this basic result for problem  $P'(e)$ . (Propositions 3.8-3.10). Proposition 3.8 was obtained previously by Oettli (1981) who independently introduced the concept of a quasiconvex map. Next, several results provide sufficient conditions for strict and semistrict quasiconvexity (Propositions 3.11-3.17). A new notion of a strictly quasiconvex map is utilized in some of these results. Finally, we obtain results on uniform and homogeneous quasiconvexity of  $f^*$  (Propositions 3.18-3.21).

Section 4 contains results on quasiconvexity and quasimonotonicity of  $f^*$ . All of these results appear to be original. The main result providing sufficient conditions for quasiconvexity of  $f^*$  is given in Proposition 4.1 and is subsequently strengthened in Propositions 4.3 and 4.4. Using the introduced notion of a quasiconcave map we are able to obtain a stronger result for the problem  $P'(e)$  (Proposition 4.6).

Another interesting result for the same problem is given in Proposition 4.7 and subsequently applied to prove a known result from quasiconvex programming (Proposition 4.8). A simple result for the special problem  $P_1(e)$  is also obtained (Proposition 4.9). Several results are proved next on strict quasiconcavity of  $f^*$  (Propositions 4.11-4.14). Proposition 4.13 utilizes a new notion of a strictly quasiconcave map. Results on uniform and homogeneous quasiconcavity (Propositions 4.15-4.20) complete the first part of this section.

The second part begins with the basic results, giving conditions for quasimonotonicity of  $f^*$  (Propositions 4.21 and 4.22). The new notions of quasimonotonic and hull quasimonotonic maps lead to a stronger result for the problem  $P^*(e)$  (Proposition 4.23). Several results on strict quasimonotonicity are given next (Propositions 4.24-4.26), using the introduced notions of strictly affine and strictly quasimonotonic maps.

In Section 5 we consider pseudoconvexity and pseudoconcavity properties of  $f^*$ . Again, all of the results appear to be new. The basic result on pseudoconvexity of  $f^*$  is stated in Proposition 5.2. Two results on strict pseudoconvexity follow next (Propositions 5.3 and 5.4). For the pseudoconcave case we do not obtain any similar results. We give several examples exhibiting this fact. Despite these examples, however, there remain several open questions concerning results in this section. Further investigation of these results may very well lead to interesting new results or improvements in the existing ones.

REFERENCES

- [1] Avriel, M., W. E. Diewert, S. Schaible and W. T. Ziemba (1981).  
Introduction to concave and generalized concave functions.  
In Generalized Concavity in Optimization and Economics  
(S. Schaible and W. T. Ziemba, eds.), Academic Press,  
New York, pp. 21-50.
- [2] Bank, B., J. Guddat, D. Klatte, B. Kummer and K. Tammer (1982).  
Nonlinear Parametric Optimization, Akademie-Verlag, Berlin.
- [3] Berge, G. (1963). Topological Spaces, MacMillan, New York.
- [4] Brosowski, B. (1982). Parametric Semi-Infinite Optimization,  
Verlag Peter Lang, Frankfurt am Main.
- [5] Fenchel, W. (1953). Convex Cones, Sets and Functions,  
Lecture Notes, Princeton University, Princeton, N.J.
- [6] Fiacco, A. V. (1983). Introduction to Sensivity and Stability  
Analysis in Nonlinear Programming, Academic Press, New York.
- [7] Fiacco, A.V. and J. Kyparisis (1982). Convexity and concavity  
properties of the optimal value function in parametric  
nonlinear programming. Technical Paper T-471, Institute  
for Management Science and Engineering, The George  
Washington University.

- [8] Ioffe, A. D. (1979). Differentielles generalisees d'applications localement Lipschitziennes d'un espace de Banach dans un autre, C.R. Acad. Sci., 289, 637-639.
- [9] Luenberger, D. G. (1968). Quasiconvex programming, SIAM J. Appl. Math., 16(5), 1090-1095.
- [10] Mangasarian, O. L. (1965). Pseudo-convex functions, SIAM J. Control, 3(2), 281-290.
- [11] Mangasarian, O. L. (1969). Nonlinear Programming, McGraw-Hill, New York.
- [12] Martos, B. (1967). Quasi-convexity and quasi-monotonicity in nonlinear programming, Studia Sci. Math. Hungar., 2, 265-273.
- [13] Ortega, J. M. and W. C. Rheinboldt (1970). Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York.
- [14] Oettli, W. (1981). Optimality conditions involving generalized convex mappings. In Generalized Concavity in Optimization and Economics (S. Schaible and W. T. Ziemba, eds.), Academic Press, New York, pp. 21-50.
- [15] Penot, J.-P. (1982). Differentiability of relations and differential stability of perturbed optimization problems, submitted to SIAM J. Control Optimiz.

- [16] Pol'yak, B. T. (1966). Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, Soviet Math., 7, 72-75.
- [17] Ponstein, J. (1967). Seven kinds of convexity, SIAM Review, 9, 115-119.
- [18] Rockafellar, R. T. (1967). Monotone processes of convex and concave type, Memoir No. 77 of Amer. Math. Soc.
- [19] Rockafellar, R. T. (1970). Convex Analysis, Princeton University Press, Princeton, N. J.
- [20] Rockafellar, R. T. (1974). Conjugate Duality and Optimization, SIAM, Philadelphia.
- [21] Tagawa, S. (1978). Optimierung mit mengenwertige Abbildungen, Ph.D. Thesis, University Mannheim.
- [22] Thompson, W. A. and D. W. Parke (1973). Some properties of generalized concave functions, Operations Research, 21(1), 305-313.



OFFICE OF NAVAL RESEARCH  
STATISTICS AND PROBABILITY PROGRAM

BASIC DISTRIBUTION LIST  
FOR  
UNCLASSIFIED TECHNICAL REPORTS

FEBRUARY 1982

Copies	Copies
<p>Statistics and Probability Program (Code 411(SP)) Office of Naval Research Arlington, VA 22217 3</p>	<p>Navy Library National Space Technology Laboratory Attn: Navy Librarian Bay St. Louis, MS 39522 1</p>
<p>Defense Technical Information Center Cameron Station Alexandria, VA 22314 12</p>	<p>U. S. Army Research Office P.O. Box 12211 Attn: Dr. J. Chandra Research Triangle Park, NC 27706 1</p>
<p>Commanding Officer Office of Naval Research Eastern/Central Regional Office Attn: Director for Science Barnes Building 495 Summer Street Boston, MA 02210 1</p>	<p>Director National Security Agency Attn: R51, Dr. Maar Fort Meade, MD 20755 1</p>
<p>Commanding Officer Office of Naval Research Western Regional Office Attn: Dr. Richard Lau 1030 East Green Street Pasadena, CA 91101 1</p>	<p>ATAA-SL, Library U.S. Army TRADOC Systems Analysis Activity Department of the Army White Sands Missile Range, NM 88002 1</p>
<p>U. S. ONR Liaison Office - Far East Attn: Scientific Director APO San Francisco 96503 1</p>	<p>ARI Field Unit-USAREUR Attn: Library c/o ODCSPER HQ USAEREUR &amp; 7th Army APO New York 09403 1</p>
<p>Applied Mathematics Laboratory David Taylor Naval Ship Research and Development Center Attn: Mr. G. H. Gleissner Bethesda, Maryland 20084 1</p>	<p>Library, Code 1424 Naval Postgraduate School Monterey, CA 93940 1</p>
<p>Commandant of the Marine Corps (Code AX) Attn: Dr. A. L. Slafkosky Scientific Advisor Washington, DC 20380 1</p>	<p>Technical Information Division Naval Research Laboratory Washington, DC 20375 1</p>
	<p>OASD (I&amp;L), Pentagon Attn: Mr. Charles S. Smith Washington, DC 20301 1</p>

# Copies

Director  
AMSAA  
Attn: DRXSY-MP, H. Cohen  
Aberdeen Proving Ground, MD 1  
21005

Dr. Gerhard Heiche  
Naval Air Systems Command  
(NAIR 03)  
Jefferson Plaza No. 1  
Arlington, VA 20360 1

Dr. Barbara Bailar  
Associate Director, Statistical  
Standards  
Bureau of Census  
Washington, DC 20233 1

Leon Slavin  
Naval Sea Systems Command  
(NSEA 05H)  
Crystal Mall #4, Rm. 129  
Washington, DC 20036 1

B. E. Clark  
RR #2, Box 647-B  
Graham, NC 27253 1

Naval Underwater Systems Center  
Attn: Dr. Derrill J. Bordelon  
Code 601  
Newport, Rhode Island 02840 1

Naval Coastal Systems Center  
Code 741  
Attn: Mr. C. M. Bennett  
Panama City, FL 32401 1

Naval Electronic Systems Command  
(NELEX 612)  
Attn: John Schuster  
National Center No. 1  
Arlington, VA 20360 1

Defense Logistics Studies  
Information Exchange  
Army Logistics Management Center  
Attn: Mr. J. Dowling  
Fort Lee, VA 23801 1

# Copies

Reliability Analysis Center (RAC)  
RADC/RBRAC  
Attn: I. L. Krulac  
Data Coordinator/  
Government Programs  
Griffiss AFB, New York 13441 1

Technical Library  
Naval Ordnance Station  
Indian Head, MD 20640 1

Library  
Naval Ocean Systems Center  
San Diego, CA 92152 1

Technical Library  
Bureau of Naval Personnel  
Department of the Navy  
Washington, DC 20370 1

Mr. Dan Leonard  
Code 8105  
Naval Ocean Systems Center  
San Diego, CA 92152 1

Dr. Alan F. Petty  
Code 7930  
Naval Research Laboratory  
Washington, DC 20375 1

Dr. M. J. Fischer  
Defense Communications Agency  
Defense Communications Engineering  
Center  
1860 Wiehle Avenue  
Reston, VA 22090 1

Mr. Jim Gates  
Code 9211  
Fleet Material Support Office  
U. S. Navy Supply Center  
Mechanicsburg, PA 17055 1

Mr. Ted Tupper  
Code M-311C  
Military Sealift Command  
Department of the Navy  
Washington, DC 20390 1

Copies

Copies

Mr. F. R. Del Priori  
Code 224  
Operational Test and Evaluation  
Force (OPTEVFOR)  
Norfolk, VA 23511

1